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OPTIMAL ESTIMATION OF EDDY VISCOSITY FOR A QUASI-THREE-DIMENSIONAL NUMERICAL TIDAL AND STORM SURGE MODEL

R. W. LARDNER AND S. K. DAS
Mathematics Department, Simon Fraser University, Burnaby, B.C., Canada

SUMMARY

It is shown that the eddy viscosity profile in a quasi-three-dimensional numerical tidal and storm surge model can be estimated by assimilation of velocity data from one or more current meters located on the same vertical line. The computational model used is a simplified version of the so-called vertical/horizontal splitting algorithm proposed by Lardner and Cekirge. We have estimated eddy viscosity both as a constant and as a variable parameter. The numerical scheme consists of a two-level leapfrog method to solve the depth-averaged equations and a generalized Crank–Nicolson scheme to compute the vertical profile of the velocity field. The cost functional in the adjoint scheme consists of two terms. The first term is a certain norm of the difference between computed and observed velocity data and the second term measures the total variation in the eddy viscosity function. The latter term is not needed when the data are exact for the model but is necessary to smooth out the instabilities associated with 'noisy' data. It is shown that a satisfactory minimization can be accomplished using either the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton algorithm or Nash's truncated Newton algorithm. Very effective estimation of eddy viscosity profiles is shown to be achieved even when the amount of data is quite small.

KEY WORDS Variational data assimilation Parameter estimation Numerical tidal model Eddy viscosity

1. INTRODUCTION

Numerical modelling has now become a major tool for the study of ocean dynamics, both in its practical aspects, such as computing the flows due to tides or storm surges, and its more theoretical and fundamental aspects. The earliest numerical models were based on the two-dimensional depth-averaged shallow water equations, which yield values of the surface elevation and depth-averaged velocity components, but these have been largely superseded in the last two decades by models using the full three-dimensional equations. Descriptions of several such models are contained in the collections edited by Heaps[1] and Nihoul and Jamart.[2]

Numerical models of flows in lakes, seas and oceans involve certain parameters, e.g. bottom friction coefficients, eddy viscosities and water depth, and certain boundary values, e.g. surface elevations on open boundaries, whose values may not be very well known. The numerical models must agree with measurements, allowing for observational errors, and the procedure generally adopted is to choose the parameters so as to minimize some cost function that measures the misfit of the computed and measured values. In such model fitting, the gradient of the cost function with respect to the parameters is needed for the optimization algorithm used in computing the input parameters that give the best fit.

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Besides this, a related problem is the study of the sensitivity of a model's outputs with respect to its input parameters and this again requires computation of the gradients of the outputs with respect to the inputs.

The variational method, first proposed by Sasaki[3,4] and Marchuk,[5] provides a powerful technique for computing such gradients which is especially useful when the number of input parameters is large. Following this method, construction of the gradient of a cost function with respect to the model parameters leads to an adjoint boundary value problem that must be solved backwards in time. Having determined the gradient, the minimization can be performed using any of a number of numerical optimization algorithms. In previous work[6,7] we have used both the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton algorithm as contained in the CONMIN subroutine of Shanno and Phua[8] and Nash's truncated Newton algorithm.[9] Descriptions of these two methods are given by Navon and Legler[10] and Nash and Nocedal.[11] We shall use them again here.

In recent years variational data assimilation has become widely used in numerical weather prediction (see e.g. reviews by Lorenc,[12] Navon[13] and LeDimet and Navon[14]) and has also been used for petroleum reservoir simulation by Chavent *et al.*[15] and groundwater flow by Carerra and Neumann.[16]

In the field of oceanography the earliest applications were made by Bennett and McIntosh[17] and Prevost and Salmon,[18] who applied the weak constraint formalism of Sasaki[4] (in which the model equations are incorporated into the cost function and so are satisfied only in a least-squares sense) to a tidal flow problem and a geostrophic flow problem respectively. Subsequent applications have employed the strong constraint formalism in which the model equations are imposed as exact constraints on the minimization; a review of this approach and an application to a model of wind-driven equatorial circulation has been given by Thacker and Long.[19] Subsequently this formalism has been used by Panchang and O'Brien[20] to determine the bottom friction coefficient in a problem of flow in a channel, using some earlier experimental results. Tziperman and Thacker[21] have used it to estimate the friction and wind forcing in ocean circulation models. Smedstad[22] and Smedstad and O'Brien[23] have extended this approach and used it to determine the effective phase speed in a model of the equatorial Pacific Ocean based on observations of sea level. Das and Lardner[6] have extended the work of Panchang and O'Brien to estimate the position-dependent drag and depth in a sectionally integrated model of flow in a channel by assimilation of periodic tidal data and have compared several minimization algorithms. Lardner[24] has used similar variational techniques to estimate the open boundary conditions in a two-dimensional tidal model and Das and Lardner[7] have extended their earlier work to the estimation of the parameters for the same two-dimensional model. This work has been applied by Lardner *et al.*[25] to obtain significant improvements in a model of tidal flow in the Arabian Gulf.

The first application of this approach to estimate viscosity was by Yu and O'Brien,[26] who estimated the eddy viscosity and surface drag coefficient in a horizontally uniform model of the ocean from measured velocities of a wind-driven flow. This work has recently been extended by Richardson and Panchang.[27] The main aim of the present paper is to incorporate such techniques into a horizontally non-uniform model of tidal and wind-driven flow, thus providing a framework for estimating eddy viscosity from data in near-coastal regions. The data in this case consist of values of water velocity obtained from current meters placed within the water body and the cost function is taken to consist basically of a norm of the difference between the computed and measured velocity values.

The adjoint methods and, in general, inverse modelling techniques are ill-posed problems. It has been found by several authors, e.g. Yeh,[28] Smedstad and O'Brien,[23] Das and Lardner[6,7] and Lardner *et al.*,[25] that if one is estimating parameters which are distributed in the space–time domain, the study is fraught with instability and non-uniqueness. We shall show that while the proposed method of estimating the viscosity function works well with exact data, such difficulties arise as soon as errors are introduced into the data. A remedy shown to be effective by Das and Lardner[6,7] was to increase the number of data stations, but this is not always a viable option. Richardson and Panchang[27] have proposed the idea of introducing a penalty term in the cost function which tends to make the parameter profiles smoothly varying. We have extended and applied this idea and found that it does provide a satisfactory way of stabilizing the estimates while leaving their essential structure intact.

In Section 2 we describe the three-dimensional numerical model that we have used. In Section 3 the corresponding discrete adjoint equations are derived and the parameter equations are set up. In Section 4 the results of several numerical tests are given and finally in Section 5 the results and conclusions are summarized.

2. THE NUMERICAL MODEL

2.1. The physical model

We use a system of co-ordinates with the z-axis pointing vertically upwards and the xy-plane occupying the undisturbed position of the water surface. The position of the bottom is taken to be $z = -h(x, y)$ while the upper surface at time t is $z = \zeta(x, y, t)$. The components of fluid velocity in the two horizontal co-ordinate directions are denoted by $u(x, y, z, t)$ and $v(x, y, z, t)$.

We consider the fluid to be incompressible and of uniform density, make the usual hydrostatic approximation to the vertical momentum equation, neglect horizontal shear stresses and use an eddy viscosity model for the vertical shears. These various approximations are usually valid and are commonly made in numerical tidal models. In the model we shall use we also neglect the advective terms in the momentum equations. These terms are generally small away from the coasts and regions of restricted flow. The continuity equation and the horizontal momentum equations then take the approximate forms

$$\zeta_t + p_x + q_y = 0, \tag{1}$$

$$u_t - (Nu_z)_z - fv + g\zeta_x = 0, \tag{2}$$

$$v_t - (Nv_z)_z + fu + g\zeta_y = 0, \tag{3}$$

where subscripts x, y, z and t are used to denote the corresponding partial derivatives and p and q are the volume fluxes defined by

$$p = \int_{-h}^{\zeta} u \, \mathrm{d}z, \qquad q = \int_{-h}^{\zeta} v \, \mathrm{d}z. \tag{4}$$

In addition, g denotes the acceleration due to gravity, $f = 2\Omega \sin \phi$ is the Coriolis parameter (where ϕ is the latitude and Ω the angular velocity of the earth's rotation) and N is the vertical eddy viscosity. In terms of p and q we define the depth-averaged components of velocity, $\bar{u}$ and $\bar{v}$, as

$$\bar{u} = p/H \approx p/h, \qquad \bar{v} = q/H \approx q/h, \tag{5}$$

where $H = h + \zeta$ is the total water depth.

There are also traction boundary conditions on the top and bottom surfaces of the water column,

$$\rho N(u_z, v_z) = (\tau^{(sx)}, \tau^{(sy)}) \quad \text{on } z = \zeta, \tag{6}$$

$$\rho N(u_z, v_z) = (\tau^{(bx)}, \tau^{(by)}) \quad \text{on } z = -h, \tag{7}$$

where ρ is the water density, $\tau^{(sx)}$ and $\tau^{(sy)}$ are the components of shear stress on the water surface caused by the wind and $\tau^{(bx)}$ and $\tau^{(by)}$ are the components of bottom friction stress.

The surface stress is assumed to be given by

$$(\tau^{(sx)}, \tau^{(sy)}) = \gamma\rho_a\sqrt{(W_x^2 + W_y^2)}(W_x, W_y), \tag{8}$$

where W_x and W_y are the components of the wind velocity, ρ_a is the air density and γ is a dimensionless friction coefficient. We shall return to the bottom friction later.

The depth-averaged momentum equations are obtained by integrating equations (2) and (3) over the water column from $z = -h$ to ζ. They take the form

$$p_t - fq + gH\zeta_x + \rho^{-1}(\tau^{(bx)} - \tau^{(sx)}) = 0, \tag{9}$$

$$q_t + fp + gH\zeta_y + \rho^{-1}(\tau^{(by)} - \tau^{(sy)}) = 0. \tag{10}$$

For the bottom friction, turbulent boundary layer models of the near-bottom flow indicate that it is physically realistic to use a quadratic dependence of bottom friction on the bottom velocity. We shall, however, make the approximation of using the depth-averaged velocity rather than the bottom velocity in this expression for the bottom friction. We therefore assume that

$$(\tau^{(bx)}, \tau^{(by)}) = \kappa\rho\sqrt{(\bar{u}^2 + \bar{v}^2)}(\bar{u}, \bar{v}) = K\rho\sqrt{(p^2 + q^2)}(p, q), \tag{11}$$

where κ is a dimensionless drag coefficient and $K = \kappa h^{-2}$. This approximation is generally made in two-dimensional hydrodynamical models. It is reasonably good for tidal flows, where in most situations the water velocity is almost uniform through the water column, but is less good for wind-driven flows.

With this approximation, equations (1), (9) and (10) form a closed system (the shallow water equations) that can be solved independently for ζ, p and q. Having solved this system, the solution can be used in the momentum equations (2) and (3) and boundary tractions (11) to determine the vertical structure of the current. This type of model is sometimes called a two-and-a-half-dimensional model. Its advantage from the point of view of estimating eddy viscosity is that eddy viscosity does not enter the shallow water system and therefore we do not require the adjoint of this system in order to estimate this parameter. This reduces the storage and CPU requirements of the computation by two or three orders of magnitude.

In addition to the differential equations, conditions are required on the lateral boundaries. On coastal boundaries the normal component of the mass flux vector is taken to be zero: $(p, q)\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit outward normal to the region. On the open part of the boundary, where the water body adjoins the open ocean, we use the most common type of boundary condition, namely the surface elevation ζ is assumed specified.

2.2. Finite difference approximations

The numerical scheme used to solve the shallow water equations (1), (9) and (10) is based on a leapfrog method with staggered grids in both space and time. An Arakawa C-grid has been used in the spatial direction and in the time direction the variables ζ and (p, q) are taken at alternating half-steps. Flat initial conditions are taken, i.e. ζ, p and q are initially zero at all grid points. The details of the numerical scheme are given in Reference 7.

At each time step the shallow water equations are first solved, then equations (2) and (3) are stepped forward. For this second stage we use a modification of the generalized Crank–Nicolson algorithm described by Lardner and Cekirge.[29] A uniform grid is introduced for the vertical direction, the bottom $z = -h$ being taken to correspond to $j = \frac{1}{2}$, while the free surface $z = \zeta$ is taken to be at $j = J + \frac{1}{2}$. The grid spacing is thus $k = H/J$. The finite difference approximations to equations (2) and (3), for the space–time grid point (i, j) corresponding to the ith time step and the jth vertical grid level, are then

$$\frac{1}{\tau}(u_{i+1,j} - u_{i,j}) - \frac{\alpha}{k^2}[N_{j+1/2}(u_{i+1,j+1} - u_{i+1,j}) - N_{j-1/2}(u_{i+1,j} - u_{i+1,j-1})]$$
$$- \frac{1-\alpha}{k^2}[N_{j+1/2}(u_{i,j+1} - u_{i,j}) - N_{j-1/2}(u_{i,j} - u_{i,j-1})] - f[\alpha v_{i+1,j} + (1-\alpha)v_{i,j}]$$
$$= -(gH\zeta_x)_{i+1/2}, \tag{12}$$

$$\frac{1}{\tau}(v_{i+1,j} - v_{i,j}) - \frac{\alpha}{k^2}[N_{j+1/2}(v_{i+1,j+1} - v_{i+1,j}) - N_{j-1/2}(v_{i+1,j} - v_{i+1,j-1})]$$
$$- \frac{1-\alpha}{k^2}[N_{j+1/2}(v_{i,j+1} - v_{i,j}) - N_{j-1/2}(v_{i,j} - v_{i,j-1})] + f[\alpha u_{i+1,j} + (1-\alpha)u_{i,j}]$$
$$= -(gH\zeta_y)_{i+1/2}. \tag{13}$$

Here $N_{j+1/2}$ denotes the eddy viscosity midway between the jth and $(j+1)$th grid points, assumed independent of time, and $u_{i,j}$ and $v_{i,j}$ are the velocity components at the jth grid point and the ith time step. The implicitness parameter α is required for stability: $\alpha = 0$ would be a fully explicit scheme and $\alpha = 0{\cdot}5$ corresponds to the Crank–Nicolson scheme. Lardner and Cekirge[29] showed that, ignoring the boundary conditions, it is necessary and sufficient for numerical stability to take $\alpha \geq 0{\cdot}5$.

The discrete approximations to the boundary conditions (6) and (7) at each time step i are taken as

$$\frac{N_s}{k}(u_{i,J+1} - u_{i,J}) = \tau_i^{(sx)}, \qquad \frac{N_s}{k}(v_{i,J+1} - v_{i,J}) = \tau_i^{(sy)}, \tag{14a,b}$$

$$\frac{N_b}{k}(u_{i,1} - u_{i,0}) = \tau_i^{(bx)}, \qquad \frac{N_b}{k}(v_{i,1} - v_{i,0}) = \tau_i^{(by)}, \tag{15a,b}$$

where $N_s = N_{J+1/2}$ and $N_b = N_{1/2}$ are the eddy viscosities at the surface and bottom respectively. Equations (12)–(15) now form a tridiagonal system that enables the updated velocities at step $i + 1$ to be readily determined from their values at step i. Flat initial values are assumed, i.e. $u_{0,j} = v_{0,j} = 0$ for all j.

3. THE ADJOINT NUMERICAL MODEL

3.1. Minimum principle

We suppose that current meters labelled by $d = 1, 2, \ldots, D$ are placed at different vertical levels at a certain point (x_D, y_D) and values of the velocities are observed for time steps

$i = I + 1, \ldots, P$. The start-up interval consisting of steps $j = 1, 2, \ldots, I$ is imposed to allow the transients arising from the initial conditions to become sufficiently small, so that the computed solution should agree with the observed values to within some tolerance in the interval $I + 1 \le j \le P$. We denote the observed velocities by $\{U_d^i, V_d^i\}$ at step i.

The meter stations may coincide with grid levels, but more generally we suppose that the corresponding computed values of velocity at station d are $\Sigma_j B_{j,d} u_{i,j}$ and $\Sigma_j B_{j,d} v_{i,j}$, where $B_{j,d}$ are appropriate interpolation coefficients and $u_{i,j}$ and $v_{i,j}$ are computed by solving equations (12)–(15) at the point (x_D, y_D). The discrepancy in the computed value at meter d and step i is then

$$\Delta U_d^i \equiv \sum_j B_{j,d} u_{i,j} - U_d^i, \qquad \Delta V_d^i \equiv \sum_j B_{j,d} v_{i,j} - V_d^i. \tag{16}$$

We suppose that the parameters in the model must be chosen so as to minimize the objective function

$$F = \tfrac{1}{2} \sum_{d=1}^{D} K_d \sum_{i=I+1}^{P} [(\Delta U_d^i)^2 + (\Delta V_d^i)^2] + \tfrac{1}{2}\beta \sum_{j=2}^{J-1} (N_{j+1/2} - N_{j-1/2})^2$$
$$+ \tfrac{1}{2}\beta'[(N_{3/2} - N_{1/2})^2 + (N_{J+1/2} - N_{J-1/2})^2], \tag{17}$$

where the quantities K_d are the respective weights given to the observational discrepancies at the different data stations. The last two terms in (17) are included to penalize large fluctuations in the estimated eddy viscosity from one level to another and are analogous to similar continuous penalty terms introduced by Richardson and Panchang.[27] They are not always needed, as we shall see, but there are situations in which they definitely are needed if the parameter estimates are to be acceptable. The reason for separating the last two terms with a different coefficient will become apparent below.

3.2. *The adjoint equations*

Introducing Lagrange multipliers, $\lambda_{i+1,j}$, $\mu_{i+1,j}$, η_i, η'_i, ε_i and ε'_i for the constraints (12)–(15), we have the following expression for the first variation of F:

$$\delta F = \sum_{d=1}^{D} \sum_{i=I+1}^{P} \left(\Delta U_d^i \sum_j B_{j,d} \delta u_{i,j} + \Delta V_d^i \sum_j B_{j,d} \delta v_{i,j} \right)$$
$$+ \beta \sum_{j=2}^{J-1} (N_{j+1/2} - N_{j-1/2})(\delta N_{j+1/2} - \delta N_{j-1/2})$$
$$+ \beta'[(N_{3/2} - N_{1/2})(\delta N_{3/2} - \delta N_{1/2}) + (N_{J+1/2} - N_{J-1/2})(\delta N_{J+1/2} - \delta N_{J-1/2})]$$
$$+ \sum_{i=0}^{P-1} \sum_{j=1}^{J} \lambda_{i+1,j} \delta(12) + \sum_{i=0}^{P-1} \sum_{j=1}^{J} \mu_{i+1,j} \delta(13)$$
$$+ \sum_{i=1}^{P} [\eta_i \delta(14a) + \eta'_i \delta(14b) + \varepsilon_i \delta(15a) + \varepsilon'_i \delta(15b)], \tag{18}$$

where $\delta(12)$ means the variation of the left side of equation (12) and so on. The right sides of these equations are independent of viscosity and so do not contribute. The sums in the last two lines of (18) are transformed using summation-by-parts formulae. In order to remove certain terms, we choose

$$\eta_i = k^{-1}[\alpha\lambda_{i,J+1} + (1-\alpha)\lambda_{i+1,J+1}], \qquad \eta_i' = k^{-1}[\alpha\mu_{i,J+1} + (1-\alpha)\mu_{i+1,J+1}],$$
$$\varepsilon_i = -k^{-1}[\alpha\lambda_{i,1} + (1-\alpha)\lambda_{i+1,1}], \qquad \varepsilon_i' = k^{-1}[\alpha\mu_{i,1} + (1-\alpha)\mu_{i+1,1}] \tag{19}$$

and also make the Lagrange multipliers satisfy the boundary and final conditions

$$\lambda_{i,0} = \lambda_{i,1}, \qquad \lambda_{i,J} = \lambda_{i,J+1}, \qquad \mu_{i,0} = \mu_{i,1}, \qquad \mu_{i,J} = \mu_{i,J+1}, \tag{20}$$

$$\lambda_{P+1,j} = 0, \qquad \mu_{P+1,j} = 0. \tag{21}$$

Then equation (18) can be written in the form

$$\delta F = \sum_{i=1}^{P}\sum_{j=1}^{J}(X_{i,j}\delta u_{i,j} + Y_{i,j}\delta v_{i,j}) + \sum_{j=0}^{J} Z_{j+1/2}\delta N_{j+1/2}, \tag{22}$$

where

$$X_{i,j} = \Theta_i \sum_{d=1}^{D} K_d B_{j,d}\Delta U_d^j + \frac{1}{\tau}(\lambda_{i,j} - \lambda_{i+1,j}) + f[\alpha\mu_{i,j} - (1-\alpha)\mu_{i+1,j}]$$
$$- \frac{\alpha}{k^2}[N_{j+1/2}(\lambda_{i,j+1} - \lambda_{i,j}) - N_{j-1/2}(\lambda_{i,j} - \lambda_{i,j-1})]$$
$$- \frac{1-\alpha}{k^2}[N_{j+1/2}(\lambda_{i+1,j+1} - \lambda_{i+1,j}) - N_{j-1/2}(\lambda_{i+1,j} - \lambda_{i+1,j-1})], \tag{23}$$

$$Y_{i,j} = \Theta_i \sum_{d=1}^{D} K_d B_{j,d}\Delta V_d^j + \frac{1}{\tau}(\mu_{i,j} - \mu_{i+1,j}) - f[\alpha\lambda_{i,j} - (1-\alpha)\lambda_{i+1,j}]$$
$$- \frac{\alpha}{k^2}[N_{j+1/2}(\mu_{i,j+1} - \mu_{i,j}) - N_{j-1/2}(\mu_{i,j} - \mu_{i,j-1})]$$
$$- \frac{1-\alpha}{k^2}[N_{j+1/2}(\mu_{i+1,j+1} - \mu_{i+1,j}) - N_{j-1/2}(\mu_{i+1,j} - \mu_{i+1,j-1})], \tag{24}$$

$$Z_{j+1/2} = \Lambda_{j+1/2} + \beta(-N_{j+3/2} + 2N_{j+1/2} - N_{j-1/2}), \quad j = 2, \ldots, J-2,$$
$$Z_{1/2} = \beta'(-N_{3/2} + N_{1/2}), \qquad Z_{3/2} = \Lambda_{3/2} + \beta(-N_{5/2} + N_{3/2}) + \beta'(N_{3/2} - N_{1/2}), \tag{25}$$
$$Z_{J+1/2} = \beta'(N_{J+1/2} - N_{J-1/2}),$$
$$Z_{J-1/2} = \Lambda_{J-1/2} + \beta(N_{J-1/2} - N_{J-3/2}) + \beta'(-N_{J+1/2} + N_{J-1/2}).$$

Here we use the notation $\Theta_i = 1$ if $I + 1 \leq i \leq P$ and $\Theta_i = 0$ otherwise and define

$$\Lambda_{j+1/2} = \sum_{i=0}^{J} \frac{1}{k^2}\{(u_{i,j+1} - u_{i,j})[\alpha(\lambda_{i,j+1} - \lambda_{i,j}) + (1-\alpha)(\lambda_{i+1,j+1} - \lambda_{i+1,j})]$$
$$+ (v_{i,j+1} - v_{i,j})[\alpha(\mu_{i,j+1} - \mu_{i,j}) + (1-\alpha)(\mu_{i+1,j+1} - \mu_{i+1,j})]\}. \tag{26}$$

To remove the first terms from equation (22), we require that $X_{i,j} = 0$ and $Y_{i,j} = 0$. Together with the boundary and final conditions (20) and (21), these two equations constitute the adjoint boundary value problem. They can be solved for $\lambda_{i,j}$ and $\mu_{i,j}$ by stepping backwards in time. At each time step the equations $X_{i,j} = 0$ and $Y_{i,j} = 0$ provide a tridiagonal system that can be solved by almost the same procedure as that for the forward problem.

Having determined $\lambda_{i,j}$ and $\mu_{i,j}$, the remaining terms in (22) provide the components of the gradient of F with respect to the parameters:

$$\frac{\partial F}{\partial N_{j+1/2}} = Z_{j+1/2}.$$

In particular, two of these components are

$$\frac{\partial F}{\partial N_{1/2}} = \beta'(N_{1/2} - N_{3/2}), \qquad \frac{\partial F}{\partial N_{J+1/2}} = \beta'(N_{J+1/2} - N_{J-1/2}). \tag{27}$$

Since these two gradients are independent of the dynamical variables u, v, λ and μ, it follows that the computed velocities are actually independent of the surface and bottom values of eddy viscosity, $N_s = N_{J+1/2}$ and $N_b = N_{1/2}$. It is immaterial what values are given these two parameters. If we choose $\beta' > 0$, it is clear from (27) that the minimization will automatically make $N_{1/2} = N_{3/2}$ and $N_{J+1/2} = N_{J-1/2}$. However, we prefer to take $\beta' = 0$ and obtain the estimates of N_s and N_b by linear extrapolation from the two nearest grid points in order to get a smoother eddy viscosity profile. This reduces the dimension of the parameter space to $J - 1$ and the remaining gradient components are

$$\frac{\partial F}{\partial N_{j+1/2}} = \Lambda_{j+1/2} + \beta(-N_{j+3/2} + 2N_{j+1/2} - N_{j-1/2}), \quad j = 2, \ldots, J-2,$$

$$\frac{\partial F}{\partial N_{3/2}} = \Lambda_{3/2} + \beta(-N_{5/2} + N_{3/2}), \qquad \frac{\partial F}{\partial N_{J-1/2}} = \Lambda_{J-1/2} + \beta(N_{J-1/2} - N_{J-3/2}). \tag{28}$$

Having determined the gradient of F, the minimization can proceed via one of several optimization algorithms. Several of these were examined earlier by Das and Lardner.[6,7] Here we have used the BFGS algorithm as contained in the CONMIN subroutine of Shanno and Phua[8] and the truncated Newton algorithm of Nash.[9]

4. NUMERICAL RESULTS

4.1. *The test model*

The region used for testing the algorithm consists of a rectangular bay with an open boundary occupying two sides. The region covers the grid points for which $2 \le m, n \le 16$, with closed boundaries along the sides $m = 1{\cdot}5$ and $n = 1{\cdot}5$ and open boundaries along the sides $m = 16$ and $n = 16$. The grid sizes are $\Delta x = \Delta y = 40{,}000$ m and the Coriolis parameter is $f = 1{\cdot}22 \times 10^{-4}$. The depth h and bottom friction coefficient K were taken to increase linearly with $m + n$ from 37 m and $0{\cdot}11 \times 10^{-5}$ m^{-2} at the closed corner (2,2) to 93 m and $0{\cdot}39 \times 10^{-5}$ m^{-2} at the open corner (16,16).

Two constituents of the tidal driving force were incorporated via the following prescribed values of water height on the open boundary:

$$\zeta(m,16) = \sin(\omega t) + [(m-2)/14]\sin(2\omega t) \quad (2 \le m \le 16),$$

$$\zeta(16,n) = [(n-2)/14]\sin(\omega t) + \sin(2\omega t) \quad (2 \le n \le 16),$$

where $\omega = 2\pi/T$ with period $T = 12$ h. In most of the tests a surface wind shear stress was assumed to be acting in the y-direction, varying sinusoidally with period 72 h and maximum amplitude equal to $1{\cdot}5\ \mathrm{N\,m^{-2}}$. The time step for the computation was 360 s, which is small enough to maintain stability as well as to provide a reasonably accurate solution.

It was assumed that current meters are located on a vertical line at the grid point (10, 10), at which point the depth is 69 m. Synthetic data from these meters were computed using the algorithm described in Section 2. The simulation was run for a total period of 96 h, with computed velocities from the final 12 h being compared with these 'observed' values for the parameter estimations.

The true eddy viscosity was assumed to have a trapezoidal form as shown in Figure 1, with various combinations of the surface, middle and bottom values N_s, N_v and N_b being used. The thicknesses of the surface and bottom layers, d_1 and d_2, were taken as fixed at 20 m. The number of vertical levels inside the water column was taken as $J = 10$ or 15 and the values of eddy viscosity were estimated at the half-integer points (see Figure 1 for the case $J = 10$), so the dimension of the parameter space was nine or 14 in the two cases.

Numerical minimization was carried out using the subroutine CONMIN[8] or Nash's truncated Newton subroutine.[9] Most of the results presented relate to the former of these two. For this subroutine it is necessary to specify a tolerance, which is the magnitude of the gradient of the objective function at which the subroutine terminates. It was found that some experimentation was needed to find the best value of this quantity in each case tested, as will be seen from the results below.

4.2. *Test of the adjoint*

Assuming that the objective function F is twice-differentiable, then, if $\mathbf{N}$ is any point in the parameter space and $\mathbf{U}$ any unit vector, the quantity

$$\Phi(\alpha) = [F(\mathbf{N} + \alpha\mathbf{U}) - F(\mathbf{N})]/(\alpha\mathbf{U}\cdot\nabla F) - 1$$

is of order α as $\alpha \to 0$. In order to test that the computation of the gradient via the adjoint system has been programmed correctly, the values of $\Phi(\alpha)$ have been computed for a given $\mathbf{N}$ and a series of decreasing values of α and for $\mathbf{U}$ in each of the co-ordinate directions.

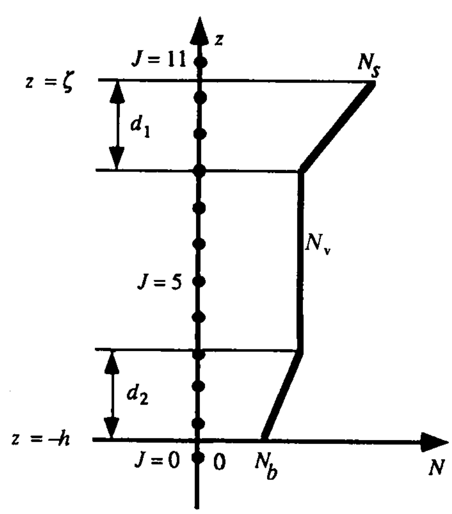

Figure 1. The assumed form of the eddy viscosity profile. Also shown are the vertical grid points in the case $J = 10$

Results for the first direction are shown in Table I: Φ decreases with α until the limit of machine accuracy is exceeded. Similar results were found for all the other directions.

4.3. Basic parameter estimation tests

In these tests it was assumed that three current meters are located at the levels $j = 2$, 5 and 8 and data from all 120 steps were assimilated. Three eddy viscosity profiles were tested: (i) constant viscosity, $N_s = N_v = N_b = 0{\cdot}065$; (ii) $N_s = N_b = 0{\cdot}02$ and $N_v = 0{\cdot}065$; (iii) $N_s = 0{\cdot}1$. $N_v = 0{\cdot}03$ and $N_b = 0{\cdot}01$.

In case (i), with a starting guess $N = 0{\cdot}05$ at all levels, convergence occurs after 73 iterations when the tolerance is set at 10^{-8} and the estimated parameters are accurate to four figures. With a tolerance of 10^{-7} convergence occurs in 51 iterations but with errors of up to 2% in the estimated values.

More detailed results for case (ii) are given in Table II, where the final estimated values of the parameters and the objective function are given for three values of the tolerance. The last column shows the true values of the parameters. Figure 2 shows graphs of the objective function and the magnitude of its gradient as functions of the number of iterations.

In Table III are given some corresponding results for case (iii) with a starting guess for eddy viscosity of 0·02 at all levels. In this case it was found that the starting value of 0·05 did not converge, because during the early iterations the CONMIN subroutine produced intermediate estimates that contained negative values of eddy viscosity at some of the levels and the

Table I. Values of Φ(α) for a sequence of decreasing values of α

α	Φ(α)
10^{-3}	$0{\cdot}25800 \times 10^{-1}$
10^{-4}	$0{\cdot}27027 \times 10^{-2}$
10^{-5}	$0{\cdot}27153 \times 10^{-3}$
10^{-6}	$0{\cdot}27165 \times 10^{-4}$
10^{-7}	$0{\cdot}27017 \times 10^{-5}$
10^{-8}	$-0{\cdot}41540 \times 10^{-6}$
10^{-9}	$-0{\cdot}36205 \times 10^{-5}$

Table II. Estimated and true values of eddy viscosity for case (ii) and three different values of the tolerance

Tolerance:	10^{-6}	10^{-7}	10^{-8}	True value of eddy viscosity
No. of iterations:	46	79	82	
Level 9·5	0·03558	0·03552	0·03553	0·03553
Level 7·5	0·06692	0·06502	0·06500	0·06500
Level 5·5	0·06696	0·06499	0·06500	0·06500
Level 3·5	0·06103	0·06499	0·06500	0·06500
Level 1·5	0·03585	0·03552	0·03553	0·03553
Objective function	$0{\cdot}399 \times 10^{-9}$	$0{\cdot}143 \times 10^{-13}$	$0{\cdot}268 \times 10^{-16}$	

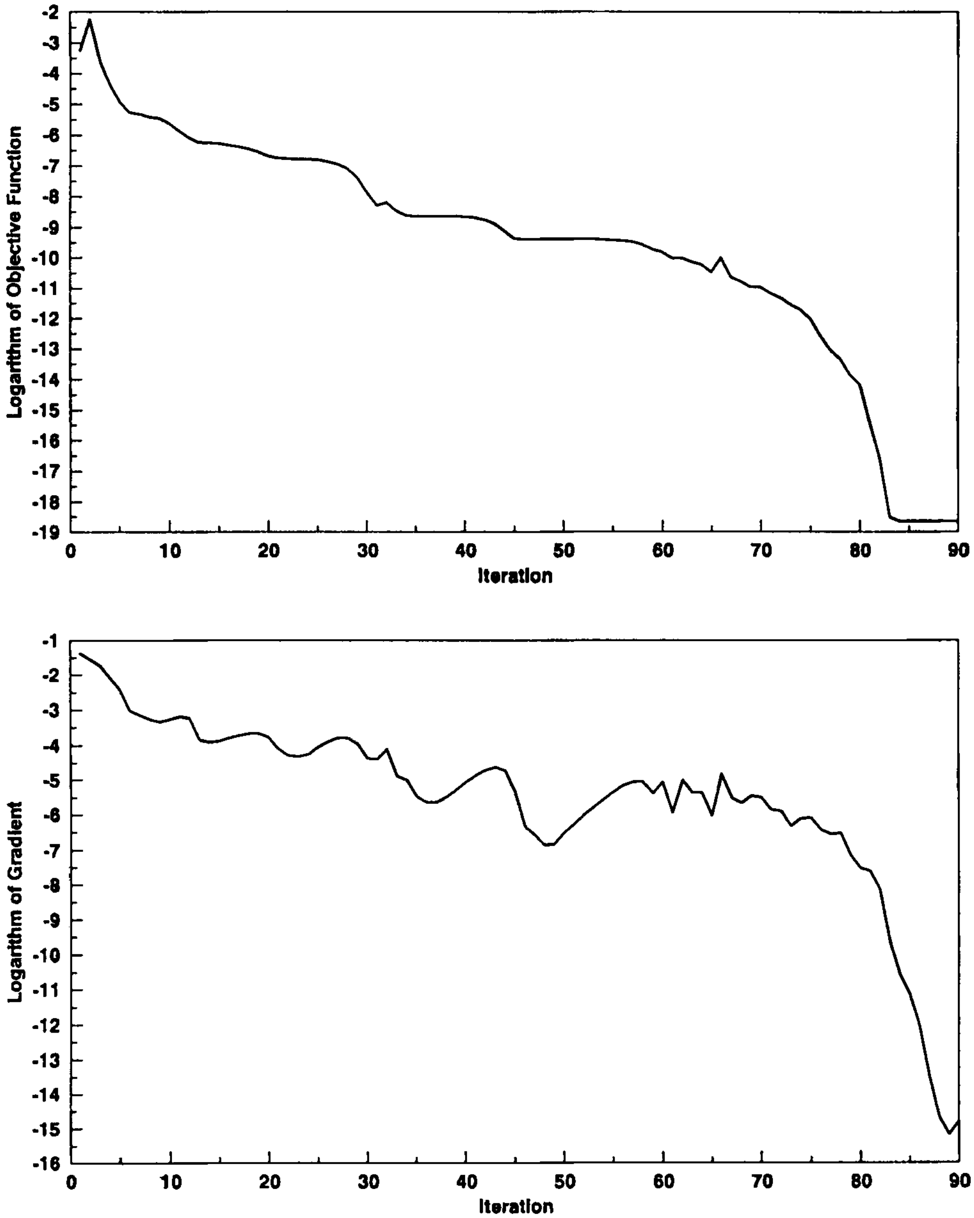

Figure 2. Logarithms of the cost function and its gradient as functions of the number of iterations

computation overflowed. This points to a general potential hazard in using an unconstrained minimization algorithm in a problem where the parameters are constrained by the physics to be positive. We have found that this difficulty can be avoided in all the cases tested by starting with an initial guess that is smaller than the true values, so that the overshoot of the early estimates is away from zero.

Richardson and Panchang[27] found that they could not obtain good estimates of eddy

Table III. Estimated and true values of eddy viscosity for case (iii) and three different values of the tolerance

Tolerance:	10^{-5}	10^{-6}	10^{-7}	True value of eddy viscosity
No. of iterations:	73	115	117	
Level 9·5	0·06760	0·07588	0·07585	0·07585
Level 7·5	0·03244	0·03000	0·03000	0·03000
Level 5·5	0·03215	0·03000	0·03000	0·03000
Level 3·5	0·02612	0·03001	0·03000	0·03000
Level 1·5	0·01723	0·01690	0·01690	0·01690
Objective function	$0{\cdot}132 \times 10^{-7}$	$0{\cdot}349 \times 10^{-12}$	$0{\cdot}164 \times 10^{-14}$	

viscosity, except at lower levels of the water column, when wind forcing is absent. Their explanation for this is that when driven by tidal forcing alone, the vertical velocity profile is almost uniform throughout the water column except very near the bottom, so the eddy viscosity terms in the momentum equations are very small and the inverse problem becomes ill-conditioned.

For this reason we have repeated the estimates of case (ii) in the case when the currents are driven by tidal forcing only. We found that good estimates are still obtained but more iterations are required. With a tolerance of 10^{-7} convergence occurs in 65 iterations but the parameter estimates have errors of as much as 5%. With a tolerance of 10^{-8} convergence occurs in 88 iterations with a maximum error of 3%. With a tolerance of 10^{-9} convergence occurs in 133 iterations and the estimates are accurate to four significant figures. When the wind forcing was also present, this level of accuracy was achieved in 82 iterations at a tolerance of 10^{-8} (see Table II). Thus, at least for the parameter values of this particular case, there is still enough vertical structure to allow estimates to be made from a tidal flow alone, though it is more expensive to do so.

4.4. *Tests with reduced data*

Next we examined the question of whether reliable estimates of eddy viscosity could be made with data from fewer than three current meters. For this purpose we used the values in case (ii) above.

With two current meters (at levels 2 and 8) convergence occurs in 46 iterations when the tolerance is 10^{-7} but the estimates have errors of up to 7%. With a tolerance of 10^{-8} convergence takes 87 iterations and the maximum error is less than 2%. In order to produce results accurate to four significant figures, a tolerance of 10^{-10} is required and convergence takes 145 iterations. Thus there is some deterioration in efficiency from Table II, but good estimates are still obtained.

For a single current meter (at level 5) some of the results are summarized in Table IV. Here the starting guess had to be reduced to $N = 0{\cdot}03$ at all levels to avoid computational overflow caused by intermediate negative viscosities. The best estimates that could be obtained contain errors of up to about 2% and this was not improved by reducing the tolerance. This may be due to the limits of machine accuracy being exceeded. However, an error of this level would certainly be acceptable in most applications and our conclusion is that even velocity data from a single level are enough to provide practical estimates of viscosity.

Secondly, the question arises as to whether data from fewer than the full 120 time steps

Table IV. Estimated and true values of eddy viscosity for case (ii) and two different values of the tolerance when data from a single level are assimilated

Tolerance:	10^{-9}	10^{-10}	True value of eddy viscosity
No. of iterations:	86	148	
Level 9·5	0·03556	0·03587	0·03553
Level 8·5	0·05182	0·05091	0·05105
Level 7·5	0·06359	0·06419	0·06500
Level 6·5	0·06469	0·06547	0·06500
Level 5·5	0·06571	0·06507	0·06500
Level 4·5	0·06593	0·06604	0·06500
Level 3·5	0·06235	0·06250	0·06500
Level 2·5	0·05296	0·05272	0·05105
Level 1·5	0·03529	0·03521	0·03553
Objective function	$0·294 \times 10^{-12}$	$0·152 \times 10^{-14}$	

would be adequate. The answer turns out to be very much in the affirmative. For example, if the computations of case (ii) are repeated but using only every tenth current data value (i.e. velocity measurements every hour), there is almost no deterioration from the results summarized in Table II. Convergence occurs in 46 iterations when the tolerance is 10^{-7} but the estimates have errors of up to 7%. With a tolerance of 10^{-8} convergence takes 79 iterations and the maximum error is ± 1 in the fourth significant digit. Estimates accurate to four significant figures are produced in 81 iterations with a tolerance of 10^{-9}.

In fact, it is possible to reduce the amount of assimilated data much more than this and still obtain good parameter estimates. If data from only two time steps (spaced 6 h apart) are used, estimates that are accurate to four figures are obtained after 126 iterations with the tolerance set at 10^{-10}. Using data from a single time step does not produce very accurate estimates, although even here the general trend of the eddy viscosity profile is obtained.

These computations have been repeated for the case of two current meters at levels 2 and 8. In this case data from two time steps are not enough, the best estimates being obtained with a tolerance of 10^{-12} and having errors of over 10%. With data from three times spaced 4 h apart, estimates with errors of less than 2% are obtained after 90 iterations when the tolerance is 10^{-10} and estimates accurate to four digits are obtained in 141 iterations when the tolerance is 10^{-11}.

4.5. Estimations with a smoothing term

In all the above studies the smoothing term was ignored by setting $\beta = 0$. As expected, the effect of giving β a positive value is to decrease the variability of the estimates, a large value of β producing an almost constant eddy viscosity profile. Figure 3 shows graphs of the estimated profiles for different values of β in case (ii) above. In each case the tolerance was chosen so that the estimates had converged to four digits. For $\beta = 10^{-4}$ the estimate is close to that given in Table II, whereas for $\beta = 10$ the profile has become almost constant.

4.6. The effect of noisy data

In all the preceding examples we have used synthetic data computed by the same numerical model for which we seek to estimate the parameters. In a real application of the method the data to be assimilated would contain observational errors and, in addition, the numerical

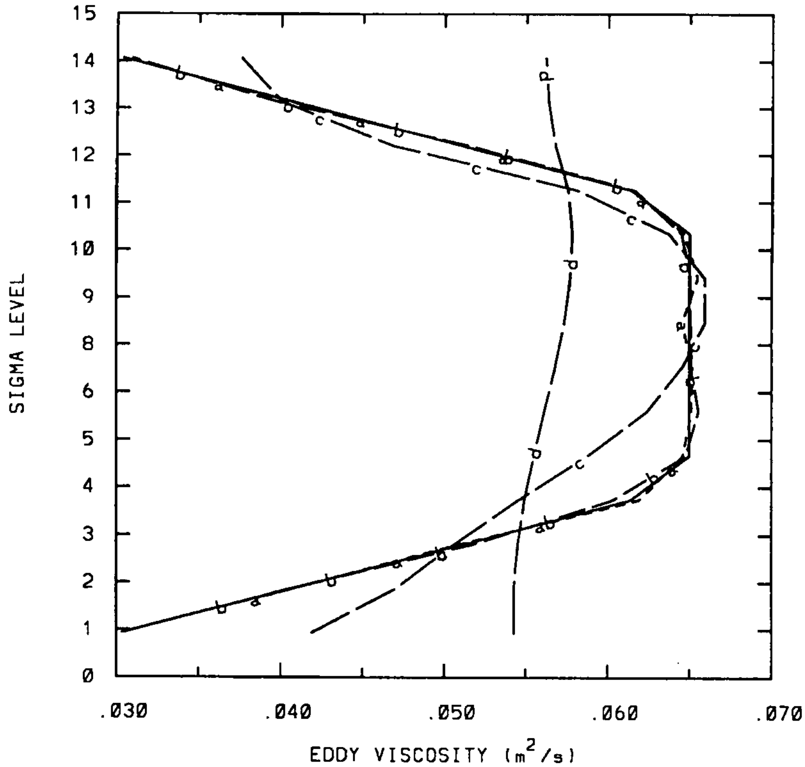

Figure 3. Computed eddy viscosity profiles for different values of the weight β. The solid line is the true profile; curve 'a' corresponds to $\beta = 0$, curve 'b' to $\beta = 0{\cdot}0001$, curve 'c' to $\beta = 0{\cdot}1$ and curve 'd' to $\beta = 10{\cdot}0$

model would be only an approximation to the physical model. In such a situation the data to be assimilated are not exactly reproducible by the model; in other words the objective function cannot be reduced to zero. In order to simulate this situation, we have attempted to estimate the parameters after deliberately introducing some random errors into the data. To do this, we have replaced each observed value $\{U_d^i, V_d^i\}$ by $\{(1 + pr_d^i)U_d^i, (1 + pr_d^i)V_d^i\}$, where r_d^i is a uniform random number lying between -1 and $+1$ and p is a factor determining the maximum percentage error.

In this particular experiment we have worked with 14 vertical levels and tested the eddy viscosity profile of case (ii) above, i.e. $N_s = N_b = 0{\cdot}02$ and $N_v = 0{\cdot}065$. The current meters were taken to be at the levels $j = 2$, 7 and 12 and data from all 120 steps were assimilated. It was found that when $\beta = 0$, the estimated parameters showed wild fluctuations from one grid level to the next. The CONMIN subroutine had difficulty terminating at smaller tolerance levels, although the parameter estimates appeared to have stabilized. The NASH subroutine had no problem converging. Also, the final estimates vary considerably from one set of random perturbations to another, but they always show similar fluctuations about the true values. For this reason we do not present any results in tabular form. The results of one simulation from CONMIN are shown as the dashed curves in Figures 4–6. These figures correspond respectively to the three cases of 2%, 4% and 8% maximum error.

It is clear from these results that even with a small error in the data it is essential to include

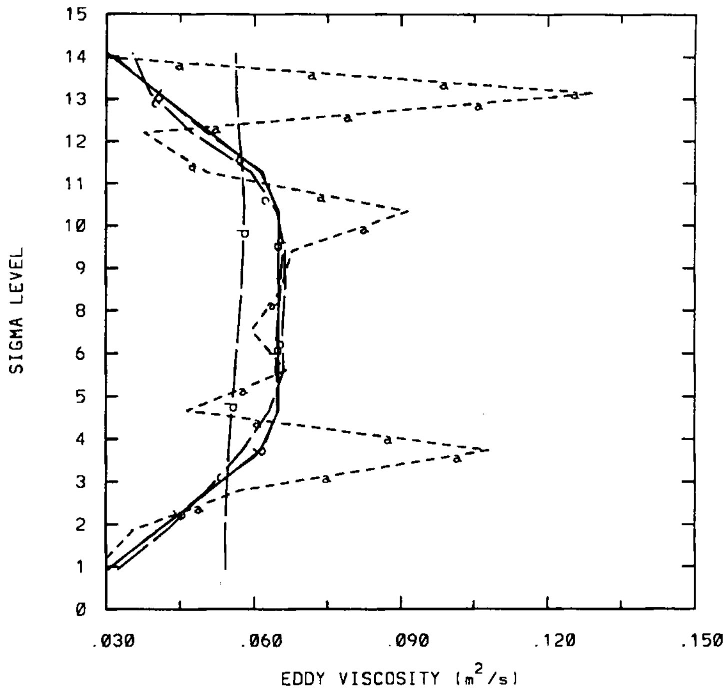

Figure 4. Computed eddy viscosity profiles for different values of the weight β when the observed data contain 2% random errors. The solid line is the true profile; curve 'a' corresponds to $\beta = 0$, curve 'b' to $\beta = 0{\cdot}0001$, curve 'c' to $\beta = 0{\cdot}01$ and curve 'd' to $\beta = 10{\cdot}0$

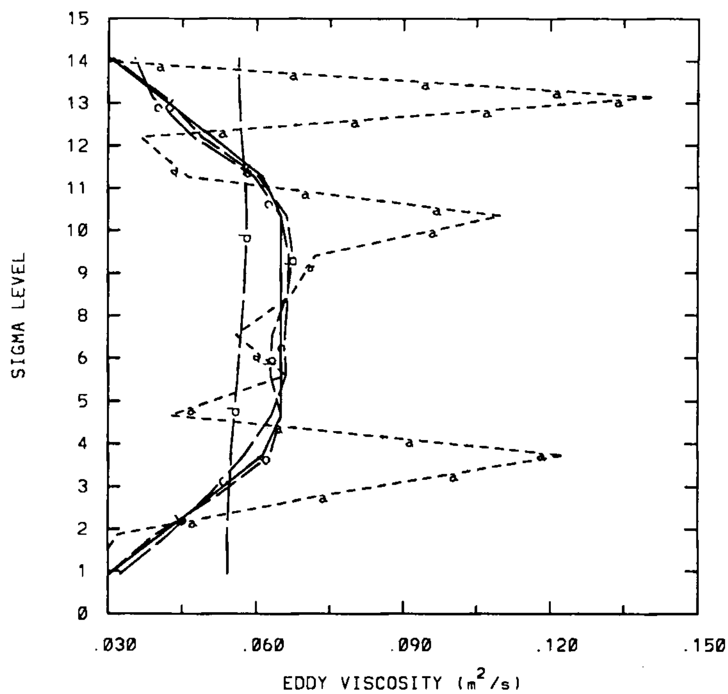

Figure 5. Same as Figure 4 but with 4% random errors

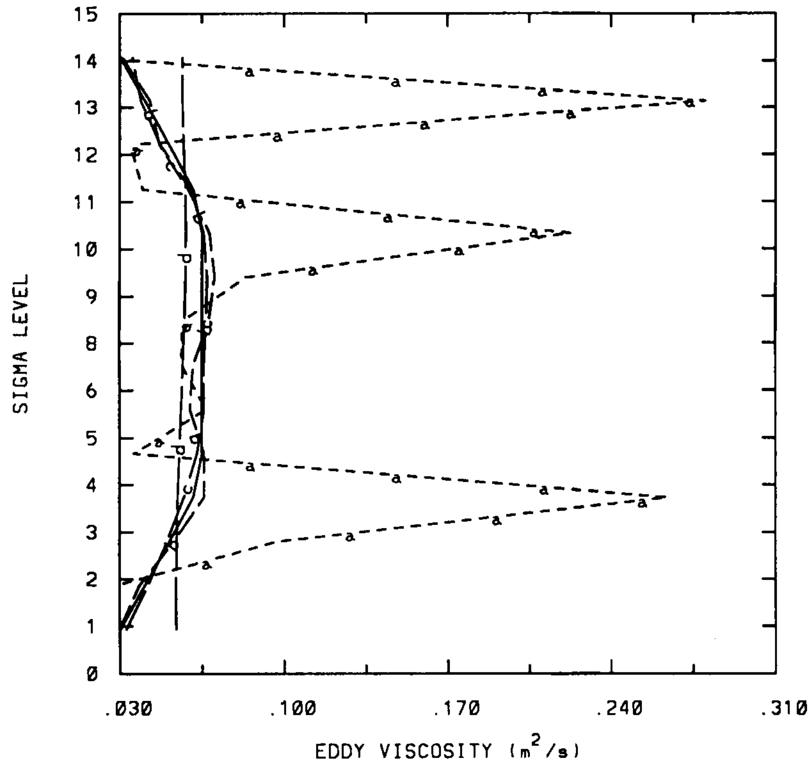

Figure 6. Same as Figure 4 but with 8% random errors

Table V. Estimates with 4% random error in the data for different values of the smoothing parameter β. Upper values refer to the CONMIN and lower values to the NASH subroutine

β:	0·0	0·0001	0·01	10·0	True viscosity
No. of iterations:	801	81	45	22	
	212	210	68	32	
Level 14·5	0·02006	0·03094	0·03538	0·05634	0·03035
	0·02813	0·03093	0·03536	0·05634	
Level 12·5	0·03634	0·04897	0·04730	0·05692	0·05105
	0·04507	0·04893	0·04732	0·05692	
Level 10·5	0·11315	0·06621	0·06498	0·05799	0·06500
	0·06688	0·06612	0·06499	0·05799	
Level 8·5	0·06712	0·06624	0·06647	0·05745	0·06500
	0·07216	0·06626	0·06646	0·05745	
Level 7·5	0·05557	0·06330	0·06589	0·05686	0·06500
	0·06003	0·06332	0·06590	0·05686	
Level 5.5	0·04174	0·06507	0·06307	0·05544	0·06500
	0·07225	0·06501	0·06307	0·05545	
Level 3·5	0·07649	0·05220	0·05030	0·05437	0·05105
	0·04720	0·05214	0·05030	0·05438	
Level 1·5	0·02672	0·03021	0·03253	0·05412	0·03035
	0·02996	0·03022	0·03254	0·05413	
Objective function	$0{\cdot}288 \times 10^{-4}$	$0{\cdot}288 \times 10^{-4}$	$0{\cdot}316 \times 10^{-4}$	$0{\cdot}188 \times 10^{-4}$	

a penalty term in the objective function to smooth the estimates. Table V shows the estimates obtained by the two subroutines for the case of 4% error when the penalty term is included. Values of β ranging between 10^{-4} and 10 are considered. In each row the top value is the estimate obtained with the CONMIN subroutine and the bottom value is that obtained with the NASH subroutine. The number of iterations for CONMIN is shown to be 801 for $\beta = 0$, which is the count after the programme was stopped mechanically. In this case the two subroutines converge to different estimates, but as β increases, the two estimates become identical.

The results are shown graphically for all three percentages in Figures 4–6. Again a large value of β is seen to produce an almost constant function, but there are intermediate values of β which are sufficiently large to smooth out the fluctuations but not so large as to eliminate the essential structure of the viscosity profile.

5. DISCUSSION

In this paper we have examined the feasibility of estimating the vertical eddy viscosity profile by assimilating velocity data from one or more current meters located on the same vertical line. In the test problem considered, the data consisted of velocity values measured over a 12 h period at one, two or three levels at a point near the centre of a rectangular bay. Synthetic data were constructed from an exact numerical solution. Two packages have been used for the numerical optimization: the BFGS method contained in the CONMIN programme and Nash's truncated Newton package. Both programmes performed equally successfully, though the NASH programme seems to work better in the presence of random errors in the data. This indicates that it may be more suitable when working with real data.

The eddy viscosity was estimated in the presence of both tidal and wind forcing. In contrast with the results found by Richardson and Panchang,[27] good estimates are obtained in the presence of tidal forcing alone, albeit at somewhat greater computational expense than when wind forcing is also present. We have obtained very accurate results when there were three data stations. Tests have shown that reliable estimates can be still be obtained with data from two current meters and even with a single meter estimates are found to be possible with errors of about 2%. Data from just two time steps spaced 6 h apart were found to be adequate to give good estimates when three current meters were used, while with two meters a minimum requirement is data from three time steps spaced 4 h apart.

The influence of noisy data on the eddy viscosity estimation has been explored. In the presence of even a small percentage error the estimated values showed wild fluctuations from point to point. It has been found, however, that the inclusion of a smoothing term in the cost function in order to penalize such fluctuations, as suggested by Richardson and Panchang,[27] leads to estimates that are reasonably smooth and still retain the essential structure of the true eddy viscosity functions. Some experimentation is necessary in each case to determine the best weight for such a penalty term. While in practice some judgement is needed in selecting this weight, the results indicate that the variational method offers a practically useful technique for estimating eddy viscosity from velocity data.

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